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The Sombor Index of a Power Graph for Some Finite Groups and Their Sombor Polynomial

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Abstract

Sombor index is a new geometric background of graph invariants and is also called a valency-based topological descriptor. It is computed by taking the radical of the sum of the squared degrees of two adjacent vertices within a graph. The Sombor polynomial also involves the degrees of two adjacent vertices where its first order derivative at x is one, is equal to the Sombor index. Meanwhile, in a power graph, two different vertices are connected by an edge if and only if one is the power of the other. The graph's vertex set consists of all the elements in a group. In this study, the Sombor index and Sombor polynomial of the power graph for some finite non-abelian groups are determined by using their definitions. The dihedral, generalized quaternion, and quasi-dihedral groups are considered. The generalization of the power graph for the quasi-dihedral groups are also found.

Keywords: Sombor index; Sombor polynomial; power graph; graph theory; group theory.

1 Introduction

Topological indices are numerical values that reveal the important details related to the connectivity or topological structure of a molecular network. They have a vital function in different areas of chemistry and related disciplines, including drug design [3], chemical property prediction, quantitative structure-activity relationship (QSAR) and quantitative structure-property relationship (QSPR) studies [10]. One of the most well-known topological indices is the Wiener index, also known as the Wiener number or the sum of all pairwise distances between atoms in a molecule [25]. The Wiener index reflects the overall branching and connectivity of a molecule and has applications in studying molecular reactivity, boiling points, and chromatographic retention times. In addition, other significant topological indices include the Randić index [11, 14], Zagreb indices [21, 27], Balaban index [9], and eccentric connectivity index [17, 20]. These indices capture different aspects of a molecular graph's structure, such as atom degree, atom connectivity, and the number of cycles. They can be used to predict physicochemical properties, biological activities, and other molecular properties [15].

Applications for topological indices have been explored in environmental chemistry, drug development, toxicology, and materials research.They provide insightful information on the connection between molecular structure and characteristics, assisting scientists in developing and refining novel molecules [4]. In the discipline of chemistry, molecules can be understood as graphs derived from graph theory, in which the chemical bonds are represented by edges, and atoms by vertices. The structural representation of molecules is provided through graphs that are also called molecular graphs.

The research on topological indices attracts much attention among chemists and mathematicians. In the past decades, researchers focused on developing new types of topological indices, approximating some molecules' physical and chemical properties and determining their correlations using statistical methods. Recently, the research has started to evolve the topological index of graphs in general. However, the graphs related to groups are important in understanding their algebraic and symmetrical properties, in which the properties are very useful in studying the properties of molecular graphs in chemistry.

In this paper, a recently developed topological index, namely the Sombor index is determined for a power graph associated to some finite non-abelian groups, specifically the dihedral groups, generalized quaternion groups and quasi-dihedral groups. Then, the Sombor polynomials of the power graph for these groups are also found. Some properties and generalization of the power graph are presented in the next section.

2 Literature Review

Some definitions and fundamental ideas pertaining to the purview of this study are provided in this section. This research involves three fields in mathematics namely topological index, graph theory, and group theory. Also stated are the preliminary results that will be utilized to proof the primary theorems. This paper focuses on a geometric approach for a topological index based on degree, which is the Sombor index [13], defined as follows:

$$SO(\Gamma) = \sum_{r,s\in\epsilon(\Gamma)} \sqrt{\sigma(r)^2 + \sigma(s)^2},$$

where Γ is a simple connected graph, $\epsilon(\Gamma)$ is the edges in graph Γ, $\sigma(r)$ and $\sigma(s)$ are the degree of vertices r and s, respectively.

Gutman [13] established the Sombor index by applying the geometric method to the degree-based topological indices. Cruz et al. [8] have provided a characterization of the Sombor index of extremal graphs, including the chemical graphs, chemical trees, and hexagonal systems. Then, Mohammadi et al. [19] have discovered that the molecular masses of the alkane, alkyl, and annulene series and the Sombor index have a strong correlation. In 2022, Oboudi [22] has studied on connected non-semiregular bipartite graphs with integer Sombor index and the infinite number of connected 3-degree bipartite graph with integer Sombor index. Gowtham and Hussin [12] extended the research to reverse Sombor index for Bistar and Corona product of graph. Recently, the Sombor polynomial of a graph has been established by [16], where the degree of polynomial is always an integer. Its definition is stated as follows:

Definition 2.1. [*16*] ***Sombor Polynomial***
Let Γ be a simple connected graph. The Sombor polynomial of Γ is defined as,

$$SO(\Gamma;x) = \sum_{r,s\in\epsilon(\Gamma} \frac{1}{\sqrt{\sigma(r)^2+\sigma(s)^2}} x^{\sigma(r)^2+\sigma(s)^2},$$

where $\sigma(r)$ and $\sigma(s)$ are the degree of the vertex r and s, respectively.

In graph theory, graph is made up of a set of vertices and a set of edges, where the edge connects the two vertices. Various types of graphs have been developed which include the power graph, as defined in Definition 2.3. Some propositions on graph theory are presented in the following to prove the main theorems.

Definition 2.2. [*24*] ***Complete Graph***
An undirected graph that has a unique edge connecting each pair of different vertices is said to be complete.

Definition 2.3. [*5*] ***Power Graph***
A power graph is an undirected graph where two distinct vertices r and s are adjacent if and only if $r^i = s$ or $s^i = r$, where $i \in \mathbb{Z}^+$.

A significant amount of work has been done on power graph research in the past decade. In 2021, Kumar et al. [18] have provided a survey on the relation of power graph with some other connected graphs such as Cayley graph. In addition, the power graphs of the finite non-abelian groups of order up to 14 which include the symmetric and dihedral groups have been determined in [6]. Recently, Ali et al. [1] found the general Randić and Harary indices of the power graphs of finite cyclic and non-cyclic groups of order pq, dihedral and generalized quaternion groups. Other types of graphs that have been receiving significant attention recently are the commuting graph of finite groups, the non-commuting graph of finite groups [23], and the zero-divisor graph of certain commutative rings [26].

3 The Vertex Degree of Power Graphs

This section presents the generalization of the power graph associated to some finite groups. Then, their vertex degree, denoted as σ, are found and stated in propositions and theorems.

Proposition 3.1. [7] *Let $H = \langle a\rangle$ be a subgroup of the dihedral group $D_{2\alpha}$, where integer $\alpha \geq 3$. The group representation is $D_{2\alpha} \cong \langle a^\alpha = b^2 = 1, bab = a^{-1}\rangle$. Suppose that $K(H)$ is the complete graph* *associated with subgroup H. Then, the power graph of $D_{2\alpha}$, denoted as $\Gamma(D_{2\alpha})$ can be illustrated as in Figure 1.*

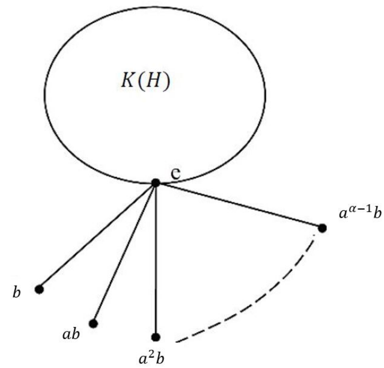

Figure 1: A power graph of $D_{2\alpha}$.

Proposition 3.2. [2] *The vertex degree in the power graph of $D_{2\alpha}$, where integer $\alpha \geq 3$ is,*

1. $\sigma(e) = 2\alpha - 1$,
2. $\sigma(a^i) = \alpha - 1, 1 \leq i \leq \alpha - 1$,
3. $\sigma(a^i b) = 1, 0 \leq i \leq \alpha - 1$.

Proposition 3.3. [7] *Let $H = \langle a \rangle$ be a subgroup of the generalized quaternion group $Q_{4\alpha}$, where integer $\alpha \geq 2$. The group representation is $Q_{4\alpha} \cong \langle a^\alpha = b^2, a^{2\alpha} = b^4 = 1, bab = a^{-1} \rangle$. Suppose that $K(H)$ is the complete graph associated with the subgroup H. Then, the power graph of $Q_{4\alpha}$, denoted as $\Gamma(Q_{4\alpha})$ can be illustrated as in Figure 2.*

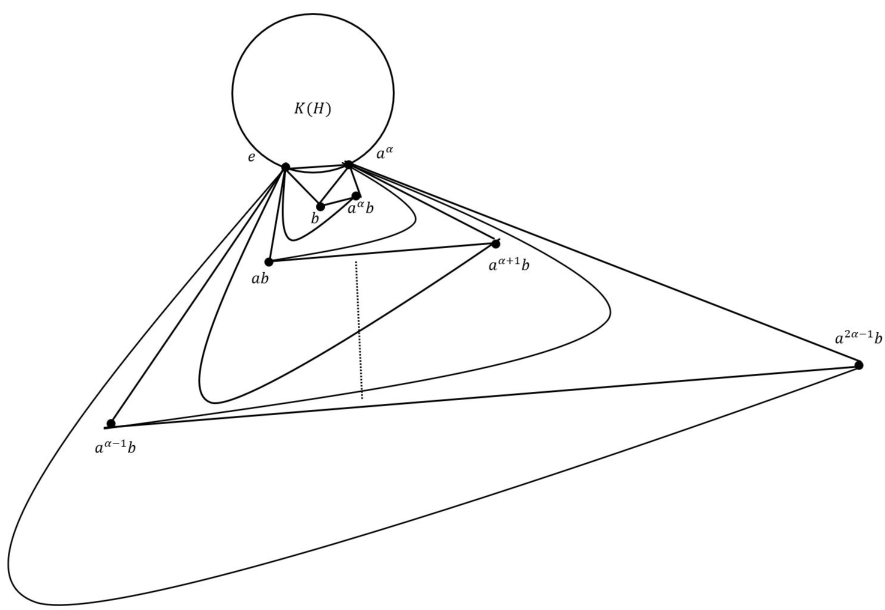

Figure 2: A power graph of $Q_{4\alpha}$.

Proposition 3.4. *The vertex degree in the power graph of $Q_{4\alpha}$, where integer $\alpha \geq 2$ is,*

1. $\sigma(e) = deg(a^{\alpha}) = 4\alpha - 1$,
2. $\sigma(a^i \backslash a^{\alpha}) = 2\alpha - 1, 1 \leq i \leq 2\alpha - 1$,
3. $\sigma(a^i b) = 3, 0 \leq i \leq 2\alpha - 1$.

Proof. The generalized quaternion group of order $4\alpha, \alpha \geq 2$, consists of two generators a and b, where $a^{\alpha} = b^2, a^{2\alpha} = b^4 = e$ and it satisfies the relation $bab = a^{-1}$. Since the element e and a^{α} are the power of all elements in $Q_{4\alpha}$, then it is adjacent to all other elements.

Hence, $deg(e) = deg(a^{\alpha}) = 4\alpha - 1$. Meanwhile, elements a^i for $1 \leq i \leq 2\alpha - 1$ are adjacent to each other since they are cyclic, $\langle a \rangle$. So that, $deg(a^i \backslash a^{\alpha}) = 2\alpha - 1$. Lastly, $\sigma(a^i b) = 3, 0 \leq i \leq 2\alpha - 1$ since $a^i b$ is the power of $\{e\}, \{a^{\alpha}\}$ and $\{a^{i+2}b\}$. □

Proposition 3.5. *Let Γ be a power graph of the generalized quaternion groups, $Q_{4\alpha}$. The number of edges in Γ, where $\alpha \geq 2$ is $|\epsilon(\Gamma)| = \alpha(2\alpha - 1) + 5\alpha$.*

Proof. Based on Figure 2, the power graph of $Q_{4\alpha}$ is made up of a complete graph for elements $a^i, 1 \leq i \leq 2\alpha - 1$. By Proposition 3.1, the total edges of that complete graph is $\dfrac{2\alpha(2\alpha - 1)}{2}$. Then, $\{e\}$ and $\{a^{\alpha}\}$ connects with another 2α elements. Meanwhile, $\{a^i b\}$ is adjacent to $\{a^{i+2}b\}$ for $0 \leq i \leq 2\alpha - 1$. Hence, $|\epsilon(\Gamma)| = \dfrac{2\alpha(2\alpha - 1)}{2} + 2\alpha + 2\alpha + \alpha = \alpha(2\alpha - 1) + 5\alpha$. □

Proposition 3.6. *Let $H = \langle a \rangle$ be a subgroup of the quasi-dihedral group $QD_{2^{\alpha}}$, where integer $\alpha \geq 4$, with group representation $QD_{2^{\alpha}} \cong \langle a, b | a^{2^{(\alpha-1)}} = b^2 = e, bab = a^{-1} \rangle$. Suppose that $K(H)$ is the complete graph associated with the subgroup H. Then, the power graph of $QD_{2^{\alpha}}$, denote as $\Gamma(QD_{2^{\alpha}})$ is depicted as in Figure 3.*

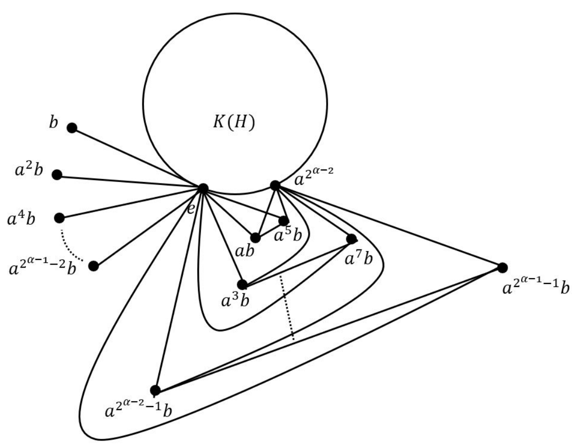

Figure 3: A power graph of $QD_{2^{\alpha}}$.

Proof. All vertices of $\Gamma(QD_{2^{\alpha}})$ are adjacent to the identity element e of $QD_{2^{\alpha}}$. Since a^i is the power of one another, then they are adjacent to each other for $1 \leq i \leq 2^{\alpha-1} - 1$. Then, $a^{2^{\alpha-2}}$ is adjacent to $a^r b$, where r is odd number up to $2^{\alpha-1} - 1$. Lastly, there is an edge connecting two vertices $a^l b$

and $a^s b$, where $3 \leq l \leq 2^{\alpha-2} - 1$ and $5 \leq s \leq 2^{(\alpha-1)} - 1$, and both l and s must be odd number. Therefore, the power graph of QD_{2^α} is displayed in Figure 3. □

Proposition 3.7. *The vertex degree in the power graph of QD_{2^α}, where $\alpha \geq 4$, is,*

1. $\sigma(e) = 2^\alpha - 1$,
2. $\sigma(a^i \backslash a^{2^{\alpha-2}}) = 2^{\alpha-1} - 1, 1 \leq i \leq 2^\alpha - 1$,
3. $\sigma(a^i b) = 1, i = 0$ *and* $1 \leq i \leq 2^\alpha - 1$, *where i is even,*
4. $\sigma(a^j b) = 3, 1 \leq i \leq 2^\alpha - 1$, *where j is odd,*
5. $\sigma(a^{2^{\alpha-2}}) = 3(2^{\alpha-2}) - 1$.

Proof. The quasi-dihedral group of order 2^α, where $\alpha \geq 4$, consists of two generators a and b, in which $a^{2^{\alpha-1}} = b^2 = e$ and it satisfies the relation $bab = a^{2^{\alpha-2}-1}$. Since an element e is the power of all elements in QD_{2^α}, then it is adjacent to all other elements. Hence, $\sigma(e) = 2^\alpha - 1$. Meanwhile, elements a^i for $1 \leq i \leq 2^\alpha - 1$ are adjacent to each other since they are cyclic, $\langle a \rangle$. So that, $\sigma(a^i \backslash a^{2^{\alpha-2}}) = 2^{\alpha-1} - 1$. $\sigma(a^i b) = 1$ for $i = 0$ and $1 \leq i \leq 2^\alpha - 1$, where i is even, since $\{e\}$ is the only power for $\{a^i b\}$, while $\sigma(a^j b) = 3$ for $1 \leq j \leq 2^\alpha - 1$, where j is odd, since $\{a^j b\}$ is the power of $\{e\}$, $\{a^{2^{\alpha-2}}\}$ and $\{a^{j+4} b\}$. Lastly, $\sigma(a^{2^{\alpha-2}}) = 3(2^{\alpha-2}) - 1$ since the element $a^{2^{\alpha-2}}$ is adjacent to all vertices except $a^i b$, where i is even. □

4 Sombor Index of the Power Graph for Finite Groups

In this section, the main results on the Sombor index of the power graph associated to the three finite non-abelian groups are stated. The Sombor index is denoted as SO while the power graph is denoted as PO. A graphical presentation of their Sombor index values is shown in Figure 4.

Theorem 4.1. *Let G be $D_{2\alpha}, \alpha \geq 3$. Then,*

$$SO(PO_G) = (\alpha - 1)\sqrt{\alpha(5\alpha - 6) + 2} + \sqrt{2}\alpha\sqrt{2\alpha(\alpha - 1) + 1} + \frac{\sqrt{2}}{2}(\alpha - 1)^2(\alpha - 2).$$

Proof. Based on Figure 1 and Proposition 3.2, there are $\alpha - 1$ edges which degree $2\alpha - 1$ and $\alpha - 1$ vertices are adjacent. α edges that have degree $2\alpha - 1$ and 1 vertices are adjacent, and $(\alpha - 1)\left(\frac{\alpha}{2} - 1\right)$ edges that connects vertices of degrees $\alpha - 1$. Therefore, by the definition of SO,

$$\begin{aligned}
SO(PO_G) &= \sum_{r,s \in \epsilon(\Gamma_G)} \sqrt{\sigma(r)^2 + \sigma(s)^2} \\
&= (\alpha - 1)\sqrt{(2\alpha - 1)^2 + (\alpha - 1)^2} + \alpha\sqrt{(2\alpha - 1)^2 + 1^2} \\
&\quad + (\alpha - 1)\left(\frac{\alpha}{2} - 1\right)\sqrt{(\alpha - 1)^2 + (\alpha - 1)^2} \\
&= (\alpha - 1)\sqrt{\alpha(5\alpha - 6) + 2} + \sqrt{2}\alpha\sqrt{2\alpha(\alpha - 1) + 1} + \frac{\sqrt{2}}{2}(\alpha - 1)^2(\alpha - 2).
\end{aligned}$$

□

Theorem 4.2. *Let G be $Q_{4\alpha}, \alpha \geq 2$. Then,*

$$SO(PO_G) = 4\sqrt{2}\left[(\alpha-1)\sqrt{2\alpha(5\alpha-3)+1} + \alpha\sqrt{4\alpha(2\alpha-1)+5}\right] + 2\sqrt{2}(2\alpha^2(\alpha-3)+9\alpha-2).$$

Proof. Based on Figure 2 and Proposition 3.4, there are degree $4\alpha-1$ and $2\alpha-1$ vertices are connected by $2(2\alpha-2)$ edges. Then, two vertices of degrees $4\alpha-1$ are connected by an edge. $(\alpha-1)(2\alpha-3)$ edges connect two vertices of degrees $2\alpha-1$ and 4α edges connect two vertices of degree $4\alpha-1$ and 3. In addition, there are also α edges that connect two vertices of degrees 3. Therefore, by definition of SO,

$$\begin{aligned}
SO(PO_G) &= \sum_{r,s\in\epsilon(\Gamma_G)} \sqrt{\sigma(r)^2+\sigma(s)^2} \\
&= 2(2\alpha-2)\sqrt{(4\alpha-1)^2+(2\alpha-1)^2} + 1\sqrt{(4\alpha-1)^2+(4\alpha-1)^2} \\
&\quad + (\alpha-1)(2\alpha-3)\sqrt{(2\alpha-1)^2+(2\alpha-1)^2} + 4\alpha\sqrt{(4\alpha-1)^2+3^2} \\
&\quad + \alpha\sqrt{3^2+3^2} \\
&= 4\sqrt{2}\left[(\alpha-1)\sqrt{2\alpha(5\alpha-3)+1} + \alpha\sqrt{4\alpha(2\alpha-1)+5}\right] \\
&\quad + 2\sqrt{2}(2\alpha^2(\alpha-3)+9\alpha-2).
\end{aligned}$$

□

Theorem 4.3. *Let G be QD_{2^α}, $\alpha \geq 4$. Then,*

$$\begin{aligned}
SO(PO_G) &= (2^{\alpha-1}-2)\left[\sqrt{(2^\alpha-1)^2+(2^{\alpha-1}-1)^2} + \sqrt{(3\times 2^{\alpha-2}-1)^2+(2^{\alpha-1}-1)^2}\right] \\
&\quad + 2^{\alpha-2}\left[\sqrt{(2^\alpha-1)^2+1} + \sqrt{(2^\alpha-1)^2+9} + \sqrt{(3\times 2^{\alpha-2}-1)^2+9}\right] \\
&\quad + \sqrt{(2^\alpha-1)^2+(3\times 2^{\alpha-2}-1)^2} + \frac{\sqrt{2}}{2}(2^{\alpha-1}-3)(2^{\alpha-1}-2)(2^{\alpha-1}-1) \\
&\quad + \sqrt{2}(3)(2^{\alpha-3}).
\end{aligned}$$

Proof. Based on Figure 3 and Proposition 3.7, there are $2^{\alpha-1}-2$ edges connect the degree $2^\alpha-1$ and $2^{\alpha-1}-1$ vertices, also degree $3\times 2^{\alpha-2}-1$ and $2^{\alpha-1}-1$ vertices. Meanwhile, there are $2^{\alpha-2}$ edges that connect vertices of degree $2^\alpha-1$ and $1, 2^\alpha-1$ and 3, and also $3\times 2^{\alpha-2}-1$ and 3. Then, there is an edge that connect two vertices of $2^\alpha-1$ and $3\times 2^{\alpha-2}-1$. $2^{\alpha-3}$ edges connect two vertices of degrees 3. Lastly, there are $\frac{(2^{\alpha-1}-2)(2^{\alpha-1}-3)}{2}$ edges that connect to two vertices of degrees $2^{\alpha-1}-1$. Therefore, by definition of SO and after simplifies,

$$\begin{aligned}
SO(PO_G) &= \sum_{r,s\in\epsilon(\Gamma_G)} \sqrt{\sigma(r)^2+\sigma(s)^2} \\
&= (2^{\alpha-1}-2)\left[\sqrt{(2^\alpha-1)^2+(2^{\alpha-1}-1)^2} + \sqrt{(3\times 2^{\alpha-2}-1)^2+(2^{\alpha-1}-1)^2}\right] \\
&\quad + 2^{\alpha-2}\left[\sqrt{(2^\alpha-1)^2+1} + \sqrt{(2^\alpha-1)^2+9} + \sqrt{(3\times 2^{\alpha-2}-1)^2+9}\right] \\
&\quad + \sqrt{(2^\alpha-1)^2+(3\times 2^{\alpha-2}-1)^2} + \frac{\sqrt{2}}{2}(2^{\alpha-1}-3)(2^{\alpha-1}-2)(2^{\alpha-1}-1) \\
&\quad + \sqrt{2}(3)(2^{\alpha-3}).
\end{aligned}$$

□

The graphical presentation of the values of $SO(PO)$ for $D_{2\alpha}$ and $Q_{4\alpha}$ is shown in the following figures.

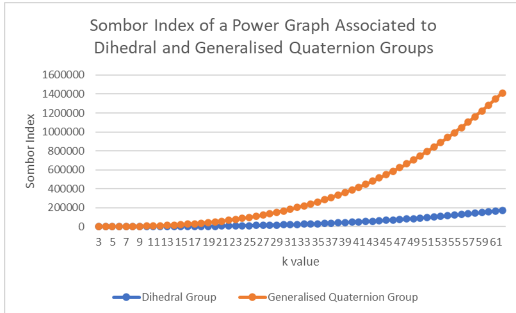

Figure 4: A graphical presentation of $SO(PO_{D_{2\alpha}})$ and $SO(PO_{Q_{4\alpha}})$.

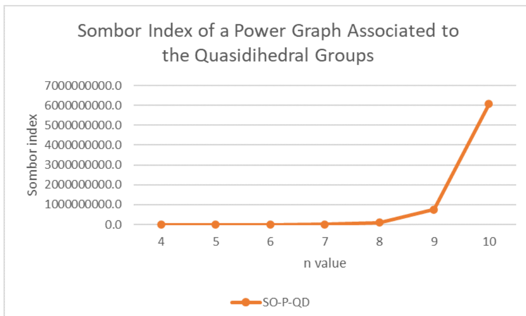

Figure 5: A graphical presentation of $SO(PO_{QD_{2^\alpha}})$.

Figures 4 shows that the Sombor index value rises with the growing order of the group, as the graph's degree increases. In addition, the same order of $D_{2\alpha}$ and $Q_{4\alpha}$ gives different complexity of power graph, where the power graph of $Q_{4\alpha}$ have more vertex degrees than $D_{2\alpha}$'s. Therefore, the $SO(PO)$ for $Q_{4\alpha}$ is higher than $D_{2\alpha}$'s. Meanwhile, in Figure 5, as the value of α increases, the $SO(PO)$ for QD_{2^α} increases.

5 Sombor Polynomial of the Power Graph for Finite Groups

This section presents the Sombor polynomial of the power graph, denoted as $SO(PO_G; x)$ for $D_{2\alpha}, Q_{4\alpha}$, and $QD_{2\alpha}$ in Theorems 5.1, 5.2 and 5.3.

Theorem 5.1. *Let G be $D_{2\alpha}$, $\alpha \geq 3$. Then,*

$$SO(PO_G; x) = \frac{\alpha - 1}{\sqrt{\alpha(5\alpha - 6) + 2}} x^{\alpha(5\alpha-6)+2} + \frac{\alpha}{\sqrt{4\alpha(\alpha - 1) + 2}} x^{4\alpha(\alpha-1)+2} + \frac{(\alpha - 1)(\alpha - 2)}{2\sqrt{2}(\alpha - 1)} x^{2(\alpha-1)^2}.$$

Proof. By Proposition 3.1 and Figure 1, the number of edges that connect two vertices are explained in the proof of Theorem 4.1. Hence, by Definition 2.1,

$$
\begin{aligned}
SO(PO_G;x) &= \sum_{r,s\in\epsilon(PO_G)} \frac{1}{\sqrt{\sigma(r)^2+\sigma(s)^2}} x^{\sigma(r)^2+\sigma(s)^2} \\
&= \frac{\alpha-1}{\sqrt{(2\alpha-1)^2+(\alpha-1)^2}} x^{(2\alpha-1)^2+(\alpha-1)^2} + \frac{\alpha}{\sqrt{(2\alpha-1)^2+1^2}} x^{(2\alpha-1)^2+1^2} \\
&\quad \frac{\frac{\alpha(\alpha-1)}{2}-(\alpha-1)}{\sqrt{(\alpha-1)^2+(\alpha-1)^2}} x^{(\alpha-1)^2+(\alpha-1)^2} \\
&= \frac{\alpha-1}{\sqrt{\alpha(5\alpha-6)+2}} x^{\alpha(5\alpha-6)+2} + \frac{\alpha}{\sqrt{4\alpha(\alpha-1)+2}} x^{4\alpha(\alpha-1)+2} \\
&\quad + \frac{(\alpha-1)(\alpha-2)}{2\sqrt{2}(\alpha-1)} x^{2(\alpha-1)^2}.
\end{aligned}
$$

□

Theorem 5.2. *Let G be $Q_{4\alpha}$, $\alpha \geq 2$. Then,*

$$
\begin{aligned}
SO(PO_G;x) =& \frac{4\alpha-4}{\sqrt{4\alpha(5\alpha-3)+2}} x^{4\alpha(5\alpha-3)+2} + \frac{4\alpha}{\sqrt{8\alpha(2\alpha-1)+10}} x^{8\alpha(2\alpha-1)+10} \\
& \frac{x^{2(4\alpha-1)^2}}{\sqrt{2}(4\alpha-1)} + \frac{(\alpha-1)(2\alpha-3)}{\sqrt{2}(2\alpha-1)} x^{2(2\alpha-1)^2} + \frac{\alpha}{3\sqrt{2}} x^{18}.
\end{aligned}
$$

Proof. By Proposition 3.3 and Figure 2, the number of edges that connect two vertices are explained in the proof of Theorem 4.2. Hence, by Definition 2.1 and the same explanation as in the proof of Theorem 4.2,

$$
\begin{aligned}
SO(PO_G;x) &= \sum_{r,s\in\epsilon(PO_G)} \frac{1}{\sqrt{\sigma(r)^2+\sigma(s)^2}} x^{\sigma(r)^2+\sigma(s)^2} \\
&= \frac{4\alpha-4}{\sqrt{4\alpha(5\alpha-3)+2}} x^{4\alpha(5\alpha-3)+2} + \frac{4\alpha}{\sqrt{8\alpha(2\alpha-1)+10}} x^{8\alpha(2\alpha-1)+10} \\
&\quad \frac{x^{2(4\alpha-1)^2}}{\sqrt{2}(4\alpha-1)} + \frac{(\alpha-1)(2\alpha-3)}{\sqrt{2}(2\alpha-1)} x^{2(2\alpha-1)^2} + \frac{\alpha}{3\sqrt{2}} x^{18}.
\end{aligned}
$$

□

Theorem 5.3. *Let G be QD_{2^α}, $\alpha \geq 4$. Then,*

$$
\begin{aligned}
SO(PO_G;x) =& (2^{\alpha-1}-2)\left[\frac{x^{5(2^{2\alpha-2})-3(2^\alpha)+2}}{\sqrt{5(2^{2\alpha-2})-3(2^\alpha)+2}} + \frac{x^{11(2^{2\alpha-2})-5(2^{\alpha-1})+2}}{\sqrt{11(2^{2\alpha-2})-5(2^{\alpha-1})+2}}\right] \\
& + 2^{\alpha-2}\left[\frac{x^{(2^\alpha-1)^2}+1}{\sqrt{(2^\alpha-1)^2+1}} + \frac{x^{(2^\alpha-1)^2}+9}{\sqrt{x^{(2^\alpha-1)^2}+9}} + \frac{x^{(3\times 2^{\alpha-2}-1)^2}}{\sqrt{3\times 2^{\alpha-2}-1)^2+9}}\right] \\
& + \frac{x^{25(2^{2\alpha-4})-72^{\alpha-1}+2}}{\sqrt{25(2^{2\alpha-4})-72^{\alpha-1}+2}} + \frac{2^{2\alpha-3}-5(2^{\alpha-2})+3}{\sqrt{2}(2^{\alpha-1}-1)} x^{2(2^{\alpha-1}-1)^2} + \frac{2^{\alpha-3}}{3\sqrt{2}} x^{18}.
\end{aligned}
$$

Proof. By Proposition 3.6 and Figure 3, the number of edges that connect two vertices are explained in the proof of Theorem 4.3. Hence, by Definition 2.1, after simplified, the $SO(PO_G)$ is as stated above. □

Next, an example of the $SO(PO)$ and $SO(PO_G; x)$ for D_8, where $\alpha = 4$ is stated in the following, by using the main results that have been obtained.

By Theorem 4.1,

$$\begin{aligned} SO(PO_G) &= (\alpha-1)\sqrt{\alpha(5\alpha-6)+2} + \sqrt{2}\alpha\sqrt{2\alpha(\alpha-1)+1} + \frac{\sqrt{2}}{2}(\alpha-1)^2(\alpha-2) \\ &= (4-1)\sqrt{4(5(4)-6)+2} + \sqrt{2}(4)\sqrt{2(4)(4-1)+1} + \frac{\sqrt{2}}{2}(4-1)^2(4-2) \\ &= 63.8595. \end{aligned}$$

By Theorem 5.1,

$$\begin{aligned} SO(PO_G; x) &= \frac{\alpha-1}{\sqrt{\alpha(5\alpha-6)+2}}x^{\alpha(5\alpha-6)+2} + \frac{\alpha}{\sqrt{4\alpha(\alpha-1)+2}}x^{4\alpha(\alpha-1)+2} \\ &\quad + \frac{\alpha(\alpha-3)+1}{2\sqrt{2}(\alpha-1)}x^{2(\alpha-1)^2} \\ &= \frac{4-1}{\sqrt{4(5(4)-6)+2}}x^{4(5(4)-6)+2} + \frac{4}{\sqrt{4(4)(4-1)+2}}x^{4(4)(4-1)+2} \\ &\quad + \frac{4(4-3)+1}{2\sqrt{2}(4-1)}x^{2(4-1)^2} \\ &= \frac{\sqrt{2}}{2}x^{58} + \frac{2\sqrt{2}}{5}x^{50} + \frac{5\sqrt{2}}{12}x^{18}. \end{aligned}$$

6 Conclusion

This paper has presented the generalization of the power graph for the quasi-dihedral groups and the general formula for the Sombor index and Sombor polynomial of the power graphs associated to some finite non-abelian groups. The results are expressed in terms of α, depending on the order of the group. The results are useful to analyze and estimate the chemical and biological molecular properties by finding their molecular graph or using point groups of certain order by checking the isomorphism between the molecular structures and the finite groups. In the future, other types of topological indices for various types of graphs can be investigated. Another new type of topological index is also possible to be developed.

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